Dynamic Optimization

An Introduction

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Outline

Background

What is Optimization?

- In general, optimization is a technique which either maximizes or minimizes the value of an objective function by systematically choosing values of inputs (or choice variables) from a feasible range.
- Optimization is one of the key ideas in the literature of Economics.

Background

What is Optimization?

- Adam Smith (1776): "It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own self-interest. We address ourselves not to their humanity but to their self-love, and never talk to them of our own necessities, but of their advantages." ***(The Wealth of Nations, Book I, Chapter II)***
- Key assumptions: (1) Rationality, and (2) Efficiency.
  - If **rationality** is assumed, firms will maximize their profits (**supply**) and households will maximize their utility (**demand**).
  - When their results are achieved, the market is **in equilibrium (efficiency).**
  - This idea is first discovered by Wilfredo Pareto (*Pareto Optimality*).

Background
Types of Optimization

- There are two general methods of optimization:
  - Analytical optimization
    - Solving the optimal solution(s) mathematically.
  - Numerical (or computational) optimization
    - Searching for the optimal solution(s) according to different algorithms (using computers).
    - For example, simulations, calibrations, and maximum likelihood estimations.

Background
Analytical Optimization

- There are three general types of analytical optimization:
  - Optimization without Constraints
    - First-order conditions (FOCs)
  - Optimization with Constraints
    - The method of Lagrangian multiplier
  - Dynamic Optimization (with/without Constraints)
    - Discrete time: The Bellman Equation
    - Continuous time: The method of Hamiltonian multiplier

Outline

EITM: The Importance of Optimization
Causality, Assumptions and Models

- Social scientists are interested in causal effects:

  How do x's affect y?

- Empirical studies (e.g., regression analysis) can show us, at most, correlations among variables (not causality!) If the coefficient on x is significant, it could imply that:
  - x causes y; or
  - y causes x ; or
  - there is another unobservable variable, called z, which contributes x and y to move simultaneously.
- But, how do we know if x's really cause y?

EITM: The Importance of Optimization
Causality, Assumptions and Models

- But, how do we know if x's really cause y?
  - NOBODY TRULY KNOWS!!
- We need to use our logical thinking and reasoning to describe why $x's$ can cause y.
  - But, the world is just too complex!
  - An easy way to do so is to build a **theoretical model** which describes some aspect of the market (or the society) that includes only those features that are needed for the propose at hand.
  - It is necessary to impose assumptions to make a model simpler.
- But how do we impose "appropriate" assumptions in a model?

EITM: The Importance of Optimization
Causality, Assumptions, and Models

A Famous Quote from Robert Solow (1956, page 65):

- "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive."
- "A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect."

EITM: The Importance of Optimization
Causality, Assumptions, and Models

- As NOBODY TRULY KNOWS how the world works, the theoretical model we build could be "wrong". In other words, the predicted results in the model can be inconsistent with what we observed in the real world.
- If this is the case, probably the assumptions we make are too sensitive (too strong) to the final results.
- Removing those assumptions / imposing some more realistic assumptions would be necessary.
- Therefore, both theoretical modeling (TM) and empirical testing (EI) enhance our understanding of the relationship between x and y.

EITM: The Importance of Optimization
Microfoundation of Macroeconomics

- In the literature of economics, we assume that people (or economic agents) are rational.
- This assumption helps us formulate human behavior in order to predict outcomes in aggregate markets. This is called the *microfoundation of macroeconomics*.
- Microfoundations refers to the microeconomic analysis of the behavior of individual agents such as households or firms that underpins a macroeconomic theory. (Barro, 1993)
- In this lecture, we study how agents face a dynamic optimization problem where actions taken in one period can affect the optimization decisions faced in future periods.

This Presentation

In this lecture, we study two methods of dynamic optimization:
(1) Discrete-time Optimization - the Bellman equations; and
(2) Continuous-time Optimization - the method of Hamiltonian multiplier.

Examples:

- Discrete-time case:
  1. Cake-eating problem
  2. Profit maximization
- Continuous-time case:
  1. Cake-eating problem
  2. Ramsey Growth Model (see lecture notes!)
- EITM: Dynamics in a Money-in-the-Utility (MIU) Model (Chari, Kehoe, and McGrattan, *Econometrica* 2000; Christiano, Eichenbaum, and Evans, *JPE* 2005)

Outline

Two-period Consumption Model

Two periods in the model
Period 1: The present; and Period 2: The future

The **two-period utility function** can be written as:

$$U = u(c_1) + \frac{1}{1+\rho} u(c_2).$$

- We call $1/(1+\rho)$ as the discount factor, where ρ is called the discount rate (or the degree of impatience).
- If an agent is more impatient ($\rho \uparrow \Rightarrow 1/(1+\rho) \downarrow$), then she would put less weight on the utility of future consumption.

Two-period Consumption Model

Two periods in the model
Period 1: The present; and Period 2: The future

The **two-period utility function** can be written as:

$$U = u(c_1) + \frac{1}{1+\rho} u(c_2).$$

Assuming that the agent has a **first-period budget constraint**:

$$Y_1 + (1+r)A_0 = c_1 + A_1,$$

- where Y_t = exogenous income at time t, A_t = assets / debts that the individual accumulates at time t, and r = an exogenous interest rate.

Two-period Consumption Model

The complete model is:

The **two-period utility function**:

$$U = u(c_1) + \frac{1}{1+\rho} u(c_2).$$

The first-period and second-period budget constraints:

$$Y_1 = c_1 + A_1 \text{ (1st-period BC), and}$$
$$Y_2 + (1+r)A_1 = c_2 \text{ (2nd-period BC).}$$

We assume that the individual does not have any inheritance/debt in period 1 (i.e., $A_0 = 0$) and does not leave any bequest/debt after period 2 (i.e., $A_2 = 0$).

Two-period Consumption Model

Lagrangian Multiplier

The system:

$$U = u(c_1) + \frac{1}{1+\rho} u(c_2).$$

and

$$Y_1 = c_1 + A_1, \text{ and } Y_2 + (1+r)A_1 = c_2.$$

To maximize the system of equations, we can apply the method of Lagrangian multiplier to solve the model:

$$L = u(c_1) + \frac{1}{1+\rho} u(c_2) + \lambda_1 (Y_1 - c_1 - A_1) + \lambda_2 (Y_2 + (1+r)A_1 - c_2),$$

where λ_1 and λ_2 are the Lagrangian multipliers.

Two-period Consumption Model

We have 5 choice variables: c_1, c_2, A_1, λ_1, and λ_2. We can solve for those variables based on the 5 first-order conditions:

$$\frac{\partial L}{\partial c_1} = 0 \Rightarrow u'(c_1) - \lambda_1 = 0 \quad (1)$$
$$\frac{\partial L}{\partial c_2} = 0 \Rightarrow \frac{1}{1+\rho} u'(c_2) - \lambda_2 = 0 \quad (2)$$
$$\frac{\partial L}{\partial A_1} = 0 \Rightarrow -\lambda_1 + (1+r)\lambda_2 = 0 \quad (3)$$
$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow Y_1 = c_1 + A_1 \quad (4)$$
$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow Y_2 + (1+r)A_1 = c_2. \quad (5)$$

Two-period Consumption Model

From equations (1)–(3), we have:

$$\lambda_1 = u'(c_1), \quad (6)$$
$$\lambda_2 = \frac{1}{1+\rho} u'(c_2), \text{ and} \quad (7)$$
$$-\lambda_1 + (1+r)\lambda_2 = 0. \quad (8)$$

Two-period Consumption Model

Now we can plug (6) and (7) into (8), we have the following equation:

$$u'(c_1) = \frac{1+r}{1+\rho} u'(c_2). \quad (9)$$

Equation (9) is called the **Euler equation**. By combining equations (4) and (5), we have the lifetime budget constraint:

$$Y_1 + \frac{Y_2}{1+r} = c_1 + \frac{c_2}{1+r}. \quad (10)$$

Finally, given a certain functional form of $u(\cdot)$, we can use equations (9) and (10) to obtain the optimal levels of c_1 and c_2, (i.e., c_1^* and c_2^*).

Two-period Consumption Model

What does the Euler equation: $u'(c_1) = \frac{1+r}{1+\rho} u'(c_2)$ tell us?

- Suppose that an agent gives up \$1 consumption today (the present), the utility cost to her will be $-u'(c_1)$. In return, she will get $\$(1+r)$ additional consumption tomorrow (the future) so that her utility gain for the tomorrow will be $(1+r)u'(c_2)$.
- However, since we assume that agents are impatient, the totally gain from giving up today's consumption for tomorrow would be $\frac{1}{1+\rho} \times (1+r)u'(c_2)$.
- In equilibrium, the agent will not give up more or less today's consumption for tomorrow at the optimal level only if $u'(c_1) = [(1+r)/(1+\rho)]u'(c_2)$.

Two-period Consumption Model: An Example

Let the utility function be the power function, $u(c) = \frac{1}{\alpha}c^{\alpha}$, where $\alpha \in (0,1)$, and $\alpha = 1/2$, we have:

$$u'(c) = c^{-1/2}. \quad (11)$$

We can plug equation (11) into equation (9), we have:

$$c_1^{-1/2} = \frac{1+r}{1+\rho} c_2^{-1/2}$$

$$\Rightarrow c_2 = \left(\frac{1+r}{1+\rho}\right)^2 c_1. \quad (12)$$

Two-period Consumption Model

An Example

Plug eq.(12) into the lifetime budget constraint (eq.(10)), we have:

$$Y_1 + \frac{Y_2}{1+r} = c_1 + \frac{1+r}{(1+\rho)^2} c_1$$

$$Y_1 + \frac{Y_2}{1+r} = c_1 \left(1 + \frac{1+r}{(1+\rho)^2}\right)$$

$$\Rightarrow c_1^* = \left(1 + \frac{1+r}{(1+\rho)^2}\right)^{-1} \left(Y_1 + \frac{Y_2}{1+r}\right). \quad (13)$$

Now we plug equation (13) into equation (12), we have:

$$c_2^* = \left(\frac{1+r}{1+\rho}\right) \left(1 + \frac{1+r}{(1+\rho)^2}\right)^{-1} \left(Y_1 + \frac{Y_2}{1+r}\right). \quad (14)$$

Two-period Consumption Model
An Example

The optimized consumption levels in period 1 and period 2 are:

$$c_1^* = \left(1 + \frac{1+r}{(1+\rho)^2}\right)^{-1} \left(Y_1 + \frac{Y_2}{1+r}\right)$$

$$c_2^* = \left(\frac{1+r}{1+\rho}\right) \left(1 + \frac{1+r}{(1+\rho)^2}\right)^{-1} \left(Y_1 + \frac{Y_2}{1+r}\right)$$

Outline

The Bellman Equation

- In the previous section, we can use the method of Lagrangian multiplier for solving a simple dynamic optimization problem. However, such the method can be sometime tedious and inefficient.
- This alternative technique is based on a recursive representation of a maximization problem, which is called the **Bellman equation**.
- The Bellman equation represents a maximization decision based on the forward (or backward) solution procedure with the property of time consistency.
- This time consistency property of the optimal solution is also known as *Bellman's optimality principle*.

Consumption Dynamics

- Let's consider the following maximization problem:

$$\max_{c_t} \left[U = \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{t-1} u(c_t) \right], \tag{15}$$

subject to the following budget constraint:

$$A_t = (1+r) A_{t-1} + Y_t - c_t. \tag{16}$$

- A_t is the **state variable** in each period t, which represents the total amount of resources available to the consumer;
- c_t is the **control variable**, where the consumer is choosing to maximize her utility. Note that c_t affects the amount of resources available for the next period, that is, A_t.

Consumption Dynamics

- For this technique of dynamic optimization, the maximum value of utility not only depends on the level of **consumption** at time t, but also the **resource left for future consumption** (i.e., A_t).
- In other word, given the existing level of asset A_{t-1}, the level of consumption chosen at time t (that is , c_t) will affect the level of assets available at time $t+1$, (that is, A_t).

Consumption Dynamics

Since the consumer would like to maximize her utility from time t onwards, we define the following value function $V_1(A_0)$ which represents the maximized value of the objective function from time $t=1$ to the last period of $t=T$:

$$V_1(A_0) = \max_{c_1} \sum_{t=1}^{T} \left(\frac{1}{1+\rho}\right)^{t-1} u(c_t) \tag{17}$$

$$V_1(A_0) = \max_{c_1} \left(u(c_1) + \left(\frac{1}{1+\rho}\right) u(c_2) + \cdots + \left(\frac{1}{1+\rho}\right)^{T-1} u(c_T) \right)$$

$$V_1(A_0) = \max_{c_1} \left\{ u(c_1) + \left(\frac{1}{1+\rho}\right) \left[\sum_{t=2}^{T} \left(\frac{1}{1+\rho}\right)^{t-2} u(c_t) \right] \right\} \tag{18}$$

subject to

$$A_t = (1+r)A_{t-1} + Y_t - c_t. \tag{19}$$

Consumption Dynamics

From equation (17), we see that $V_1(A_0)$ is the maximized value of the objective function at time $t=1$ given an initial stock of assets A_0.
After the maximization in the first period $(t=1)$, the consumer repeats the same procedure of utility maximization according to the objective function in period $t=2$, given an initial stock of assets in period 1:

$$V_2(A_1) = \max_{c_2} \sum_{t=2}^{T} \left(\frac{1}{1+\rho}\right)^{t-2} u(c_t), \tag{20}$$

subject to equation (19).
By plugging equation (20) into equation (18), we have:

$$V_1(A_0) = \max_{c_1} \left[u(c_1) + \left(\frac{1}{1+\rho}\right) V_2(A_1) \right].$$

Consumption Dynamics

The Bellman Equation

Assuming the consumer is maximizing her utility every period, we rewrite the maximization problem recursively. Therefore, we can present **the well-known *Bellman equation*** as follows:

$$V_t(A_{t-1}) = \max_{c_t} \left[u(c_t) + \left(\frac{1}{1+\rho}\right) V_{t+1}(A_t) \right],$$

where $A_t = (1+r)A_{t-1} + Y_t - c_t$.

Outline

Cake Eating Problem

Question

Suppose the size of a cake at time t is Π_t; and the utility function is presented as $u(c_t) = 2c_t^{1/2}$. Given that $\Pi_0 = 1$, $\Pi_T = 0$, and the discount rate is ρ, what is the optimal path of consumption?

Cake Eating Problem

Answer

This optimization problem can be written as:

$$\max_{c_1, c_2, \ldots, c_T} \sum_{t=1}^{T} \left(\frac{1}{1+\rho} \right)^{t-1} u(c_t), \tag{21}$$

subject to $\Pi_t = \Pi_{t-1} - c_t$, for $t = 1, 2, \ldots, T$, and $\Pi_0 = 1$ and $\Pi_T = 1$.
In this case, we see that the choice variable is c_t and the state variable Π_t.

Cake Eating Problem

The Bellman Equation

Now we can formulate the Bellman equation:

$$V(\Pi_{t-1}) = \max_{c_t} \left[u(c_t) + \frac{1}{1+\rho} V(\Pi_t) \right], \tag{22}$$

subject to

$$\Pi_t = \Pi_{t-1} - c_t. \tag{23}$$

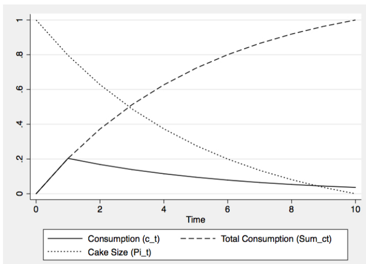
Since $u(c_t) = 2c_t^{1/2}$, we can rewrite the Bellman equation as:

$$V(\Pi_{t-1}) = \max_{c_t} \left[2c_t^{1/2} + \frac{1}{1+\rho} V(\Pi_t) \right]. \tag{24}$$

Cake Eating Problem

The Optimal Path of Cake Consumption

	A	B	C	D	E	F	G
1	Time	Consumption (c_t)	Total Consumption (Sum_ct)	Cake Size (Pi_t)		Parameters	
2	0	0	0	1			
3	1	0.203855547	0.203855547	0.796144453			
4	2	0.168475659	0.372331206	0.627668794		T	10
5	3	0.139236082	0.511567288	0.488432712		Pi_0	1
6	4	0.115071142	0.626638429	0.373361571		rho	0.1
7	5	0.095100117	0.721738547	0.278261453			
8	6	0.078595138	0.800333685	0.199666315			
9	7	0.06495466	0.865288345	0.134711655			
10	8	0.053681537	0.918969881	0.081030119			
11	9	0.044364907	0.963334788	0.036665212			
12	10	0.036665212	1	0			



Outline

Profit Maximization

Assuming that a representative firm maximizes the present value of all future profits by choosing the levels of investment I_t and labor L_t for $t = 1, 2, \ldots, T$. Therefore, we have the following maximization problem:

$$\max_{I_1, I_2, \ldots I_T, L_1, L_2, \ldots L_T} \quad \sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^t \pi_t$$

$$\max_{I_1, I_2, \ldots I_T, L_1, L_2, \ldots L_T} \quad \sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^t \left(F\left(K_t, L_t\right) - w_t L_t - I\right),$$

subject to

$$K_{t+1} = K_t - \delta K_t + I_t$$

for $t = 1, 2, \ldots, T$, and K_1 and K_T are given, π_t is the level of profit at time t, K_t is the stock of capital at time t, $F\left(K_t, L_t\right)$ is a production function, w_t is the wage rate, r and δ is the interest rate and depreciate rate in the market, respectively.

Profit Maximization

In this case,

- the choice variable is: I_t and L_t, and
- the state variable is K_t.

According to the above system, we can formulate the following Bellman equation:

$$V_t\left(K_t\right) = \max_{I_t, L_t} \left[F\left(K_t, L_t\right) - w_t L_t - I_t + \frac{1}{1+r} V_{t+1}\left(K_{t+1}\right)\right], \quad (25)$$

where

$$K_{t+1} = \left(1 - \delta\right) K_t + I_t. \quad (26)$$

Profit Maximization

In this optimization problem, we have two important conditions:

1. $\frac{\partial F(K_t, L_t)}{\partial L_t} = w_t$.
   - This result suggests that the optimal amount of labor satisfies the condition where marginal product of labor (MPL) equals the real wage rate in each period, that is $MPL_t = w_t$.
2. $\frac{\partial F}{\partial K_t} = r + \delta$.
   - This result suggests that the firm must choose a level of investment such that the marginal product of capital (MPK) equals the sum of interest rate and depreciation rate in each period, that is $MPK_t = r + \delta$.

Outline

Dynamic Optimization in Continuous Time

- In the previous sections, we study the dynamic optimization based the discrete-time dynamic model, where the change in time Δt is positive and finite (for example, we assume that $\Delta t = 1$ for all $t \geq 0$, such that t follows the sequence of $\{0, 1, 2, 3, \dots\}$.
- In the section, we consider the method of dynamic optimization in continuous time, where $\Delta t \to 0$. Therefore, we assume that agents make optimizing choices at every instant in continuous time.
- The method is called the **technique of Hamiltonian multiplier**.

Dynamic Optimization in Continuous Time

A general continuous-time maximization problem can be written as follows:

$$\max_{x_t} \int_0^T e^{-\rho t} f(x_t, A_t)\, dt, \tag{27}$$

subject to the constraint:

$$\dot{A}_t = g(x_t, A_t), \tag{28}$$

where $\dot{A}_t$ is a time derivative of A_t defined as dA_t/dt, and ρ is the discount rate in the model, which is equivalent to the ρ we use in the discrete time models.

- If $\rho = 0$, then $e^{-\rho t} = 1$. It implies that the agent does not discount the activities in the future. On the other hand, if $\rho \uparrow$, then $e^{-\rho t} \downarrow$. It implies that the agent becomes more impatient and discounts the value (or utility) of future activities more.

The Method of Hamiltonian Multiplier

We set up the following Hamiltonian function:

$$H_t = e^{-\rho t}\left[f(x_t, A_t) + \lambda_t \dot{A}_t\right], \tag{29}$$

where $\lambda_t = \mu_t e^{-\rho t}$ is called the Hamiltonian multiplier.
The three conditions for a solution:

1. The FOC with respect to the control variable (x_t):
$$\frac{\partial H_t}{\partial x_t} = 0; \tag{30}$$
2. The negative derivative of the Hamiltonian function w.r.t. A_t:
$$-\frac{\partial H_t}{\partial A_t} = \frac{d(\lambda_t e^{-\rho t})}{dt}; \tag{31}$$
3. The transversality condition:
$$\lim_{t\to\infty} \lambda_t e^{-\rho t} A_t = 0. \tag{32}$$

Outline

Cake Eating Problem

Let us consider the same cake eating problem in continuous time:

$$\max_{c_t} \int_0^T e^{-\rho t} u(c_t)\, dt, \tag{33}$$

subject to

$$\dot{\Pi}_t = -c_t, \tag{34}$$

and Π_0 and Π_1 are given. Again, the choice variable is c_t, and the state variable is Π_t.

Cake Eating Problem

We set up the Hamiltonian as follows:

$$H_t = e^{-\rho t}\left(u(c_t) + \lambda_t \dot{\Pi}_t\right), \tag{35}$$

where λ_t is the co-state variable. Now, we can plug equation (34) into equation (35), we have the final Hamiltonian equation:

$$H_t = e^{-\rho t}\left(u(c_t) - \lambda_t c_t\right) \tag{36}$$

Cake Eating Problem

We obtain the following conditions:

1. The FOC w.r.t. (with respect to) c_t:

$$\frac{\partial H_t}{\partial c_t} = e^{-\rho t}\left(u'\left(c_t\right) - \lambda_t\right) = 0$$
$$\Rightarrow \quad u'\left(c_t\right) = \lambda_t; \tag{37}$$

2. The negative derivative w.r.t. Π_t equals the time derivative of $\lambda_t e^{-\rho t}$:

$$-\frac{\partial H_t}{\partial \Pi_t} = \frac{d\left(\lambda_t e^{-\rho t}\right)}{dt}; \tag{38}$$

and

3. The transversality condition:

$$\lim_{t\to\infty} \lambda_t e^{-\rho t} \Pi_t = 0. \tag{39}$$

Outline

An MIU Model: An Introduction

- Why do people want to hold money?
  - Holding cash does not generate any returns!
- Two possible arguments:
  1. People feel good and safe if they hold some cash in their pocket (Sidrauski, 1967)
  2. Money can provide transaction services (transaction purpose!) (Baumol, 1952, Tobin, 1956)
- Here we study a basic neoclassical model where agents' utility depends directly on their consumption of goods and their holdings of money (money demand).
  - Assumption: Money yields direct utility.
- This is called the Money-in-the-Utility model (Chari, Kehoe, and McGrattan, *Econometrica* 2000; Christiano, Eichenbaum, and Evans, *JPE* 2005)

Outline

1. Background
   - What is Optimization?
   - EITM: The Importance of Optimization
2. Dynamic Optimization in Discrete Time
   - A Simple Two-period Consumption Model
   - The Bellman Equation
   - Cake Eating Problem
   - Profit Maximization
3. Dynamic Optimization in Continuous Time
   - The Method of Hamiltonian Multiplier
   - Cake Eating Problem Revisited
4. An EITM Example
   - Dynamics in a Money-in-the-Utility Model
   - TM: Theoretical Model
   - EI: Empirical Implications

An MIU Model

Consider the following Money-in-the-Utility Model:

$$W = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^{t} u\left(c_t, m_t\right),$$

where $m_t = M_t / \left(P_t N_t\right) =$ real money holding per capita in an economy. The budget constraint in the whole economy is:

$$GDP_t = Consumption_t + Investment_t + GovtSpending_t + NewNationalDebt_t + NewMoney_t.$$

We can translate as:

$$Y_t = C_t + \left[K_t - (1-\delta) K_{t-1}\right] - \tau_t N_t + \left[\frac{B}{P_t} - (1+i_{t-1}) \frac{B_{t-1}}{P_t}\right] + \left[\frac{M_t}{P_t} - \frac{M_{t-1}}{P_t}\right].$$

An MIU Model

$$Y_t = C_t + \left[K_t - (1-\delta) K_{t-1}\right] - \tau_t N_t + \left[\frac{B}{P_t} - (1+i_{t-1}) \frac{B_{t-1}}{P_t}\right] + \left[\frac{M_t}{P_t} - \frac{M_{t-1}}{P_t}\right].$$

where $Y_t =$ aggregate output (GDP), $\tau_t N_t =$ total lump-sum transfers (positive) or taxes (negative), $i =$interest rate, and $N_t =$ population.

An MIU Model

Finally, the complete system is:

$$W = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^{t} u\left(c_t, m_t\right),$$

and the budget constraint can be presented as follows:

$$Y_t + \tau_t N_t + (1-\delta) K_{t-1} + (1+i_{t-1}) \frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} = C_t + K_t + \frac{M_t}{P_t} + \frac{B_t}{P_t}.$$

We also define the production function as $Y_t = F\left(K_{t-1}, N_t\right)$. The per capita income is:

$$y_t = \frac{1}{N_t} F\left(K_{t-1}, N_t\right) = F\left(\frac{K_{t-1}}{N_t}, \frac{N_t}{N_t}\right) = F\left(\frac{K_{t-1}}{(1+n) N_{t-1}}, 1\right) = f\left(\frac{k_{t-1}}{1+n}\right),$$ where $n =$ population growth rate.

An MIU Model

Finally, the complete system is:

$$W = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^{t} u\left(c_t, m_t\right),$$

and the budget constraint (per capita) is:

$$f\left(\frac{k_{t-1}}{1+n}\right) + \tau_t + \frac{1-\delta}{1+n} k_{t-1} + \frac{(1+i_{t-1}) b_{t-1} + m_t}{(1+\pi_t)(1+n)} = c_t + k_t + m_t + b_t,$$

where π_t is the inflation rate, such that, $P_t = (1+\pi_t) P_{t-1}$, $b_t = B_t / \left(P_t N_t\right)$, and $m_t = M_t / \left(P_t N_t\right)$.

An MIU Model

We can now formulate the model as the Bellman equation:

$$V(W_t) = \max \left[u(c_t, m_t) + \frac{1}{1+\rho} V(W_{t+1}) \right],$$

subject to

$$\begin{aligned} W_{t+1} &= f\left(\frac{k_t}{1+n}\right) + \tau_{t+1} + \left(\frac{1-\delta}{1+n}\right) k_t + \frac{(1+i_t) b_t + m_t}{(1+\pi_{t+1})(1+n)} \\ &= k_{t+1} + c_{t+1} + m_{t+1} + b_{t+1}. \end{aligned}$$

An MIU Model

Finally we have the following equilibrium:

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i}{1+i}.$$

Now assume that the utility function is of the constant elasticity of substitution (CES) form:

$$u(c_t, m_t) = \left[a c_t^{1-b} + (1-a) m_t^{1-b} \right]^{1/(1-b)}.$$

The final solution based on the CES utility function is:

$$m_t = \left(\frac{a}{1-a}\right)^{-1/b} \left(\frac{i}{1+i}\right)^{-1/b} c_t, \text{ or}$$

$$\ln m_t = \frac{1}{b} \ln \left(\frac{1-a}{a}\right) + \ln c_t - \frac{1}{b} \ln \gamma,$$

where $\gamma = i/(1+i)$ = the opportunity cost of holding money.

Outline

Empirical Estimation of Money Demand

According to the theoretical solution:

$$\ln m_t = \frac{1}{b} \ln \left(\frac{1-a}{a}\right) + \ln c_t - \frac{1}{b} \ln \gamma, \tag{40}$$

where $\gamma = i/(1+i)$ which can be called the opportunity cost of holding money.
Therefore, we empirical model can be written as:

$$\ln m_t = \alpha_0 + \alpha_1 \ln c_t + \alpha_2 \ln \gamma + \varepsilon_t. \tag{41}$$

- According to equation (41), we expect that he coefficient on $\ln c_t$ is $\alpha_1 = 1$. It implies that consumption (income) elasticity of money demand is equal to 1.
- The coefficient on $\ln \frac{i}{1+i}$ is $\alpha_2 = -1/b$. For simplicity, we call it as the *interest elasticity of money demand*.

Empirical Estimation of Money Demand

The empirical result of the money demand function for the United States based on quarterly data from the period of 1984:1 – 2007:2.

Estimated Money Demand (MZM), U.S., 1984:1–2007:2

m	Const	$\ln C$	$\ln Y$	$\ln\left(\frac{i}{1+i}\right)$	m_{t-1}
1.	−8.482 (0.192)	1.357 (0.024)		−0.090 (0.010)	
2.	−10.380 (0.241)		1.500 (0.028)	−0.107 (0.010)	
3.	−0.965 (0.251)	0.153 (0.040)		−0.016 (0.004)	0.886 (0.029)
4.	−1.036 (0.275)		0.149 (0.030)	−0.016 (0.004)	0.898 (0.020)

Note: Standard errors in parentheses.

Source: Walsh (2010: 51)

$\ln m =$ the log of real money balance (M1); $c_t =$ real people consumption expenditures; and $\ln \gamma = \ln \frac{i}{1+i} =$ opportunity cost of holding money = return on M1 minus the 3-month T-Bill rate.

Thank you!

Questions?